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Math 265H September 12, 2022

- Proof of the existence of n'th roots left over from the last time.

Decimals: Let $x>0$, $x \in \mathbb{R}$

Let n_0 = largest integer $\leq x$. Having chosen $n_1, n_2, \ldots, n_{k-1}$, choose $n_k \ni$

$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \cdots + \frac{n_k}{10^k} \leq x.$$

Note that $n_i \in \{0, 1, 2, \ldots, 9\}$

Let $E = \left\{ n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k} \leq x \;\middle|\; k \in \mathbb{Z}^+ \right\}$

Then $x = \sup E$ w/ decimal expansion

$n_0 . n_1 \ldots n_k \ldots$

What can you say about positive integers that do not have 5's in their decimal expansions.

2

Complex field: $\mathbb{C}$

pairs (a,b) (ordered) $\ni$

$$(a,b)+(c,d)=(a+c,\ b+d)$$

$$(a,b)\cdot(c,d)=(ac-bd,ad+bc)$$

motivation

$$(a+ib)(c+id)=(ac-bd)+i(ad+bc),$$

where i is the "mythical" number $\ni i^2=-1$.

Theorem: $\mathbb{C}$ is a field

Proof: Check the axioms

Theorem: $(a,0)+(b,0)=(a+b,0)$

$$(a,0)\cdot(b,0)=(ab,0)$$

Proof: Straight forward

Let $i \equiv (0,1)$

definition

Theorem: $i^2 = -1$

Proof: $(0,1)\cdot(0,1) = (-1,0) = \underline{\underline{-1}}$

Theorem: $a, b \in \mathbb{R}$, then $(a,b) = a+bi$.

Proof: $a+bi = (a,0)+(b,0)(0,1)$

$= (a,0)+(0,b) = (a,b)$ ✓

Theorem: z, w complex,

$z = a+bi$; $a \equiv \underbrace{Re(z)}_{\text{real part}}$, $b = \underbrace{Im(z)}_{\text{imaginary part}}$

a) $\overline{z+w} = \bar{z} + \bar{w}$,

where $\bar{z} = a - bi$.

b) $\overline{zw} = \bar{z}\,\bar{w}$

c) $z+\bar{z} = 2Re(z)$, $z-\bar{z} = 2i\,Im(z)$

d) $z\cdot\bar{z}$ is real and positive, except when $z=0$.

④

Definition: $|z| \equiv (z \cdot \bar{z})^{\frac{1}{2}}$ positive square root

Theorem: z, w complex numbers

a) $|z| > 0$ unless $z = 0$, $|0| = 0$

b) $|\bar{z}| = |z|$

c) $|zw| = |z| \cdot |w|$

d) $|Re(z)| \leq |z|$

e) $|z+w| \leq |z| + |w|$.

Theorem: $a_1, \ldots, a_n \quad b_1, \ldots, b_n \in \mathbb{C}$

Then $\left| \sum_{j=1}^{n} a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$

Proof: $A = \sum |a_j|^2$, $B = \sum |b_j|^2$

$C = \sum a_j \bar{b}_j$

⑤

If $B=0$, $b_1 = b_2 = \dots = b_n = 0$
and the proof follows.

If $B>0$,

$$0 \le \sum |Ba_j - Cb_j|^2 = \sum (Ba_j - Cb_j)\overline{(Ba_j - Cb_j)}$$

$$= B^2 \sum |a_j|^2 - B\overline{C} \sum a_j \overline{b_j} - BC \sum \overline{a_j} b_j$$

$$+ |C|^2 \sum |b_j|^2$$

$$= B^2 A - B|C|^2 = B(AB - |C|^2)$$

It follows that $|C|^2 \le AB$, as claimed.

Euclidean space: $\mathbb{R}^K = \left\{ X = (x_1, \dots, x_K) : x_i \in \mathbb{R} \right\}$

$$x+y = (x_1+y_1, \dots, x_K+y_K)$$

$$\underbrace{\alpha}_{\text{real scalar}} X = (\alpha X_1, \alpha X_2, \dots, \alpha X_K)$$

$$\underbrace{X \cdot y \equiv \sum_{i=1}^{K} x_i y_i}_{\text{inner product}}$$

$$|X|^2 = X \cdot X$$

Theorem: $x, y, z \in \mathbb{R}^k$, α real. Then

a) $|x| \geq 0$

b) $|x| = 0$ iff $x = \vec{0}$

c) $|\alpha x| = |\alpha| |x|$

immediate

d) $|x \cdot y| \leq |x| \cdot |y|$ follows from Cauchy-Schwarz

e) $|x+y| \leq |x| + |y|$

f) $|x-z| \leq |x-y| + |y-z|$

Proof of e) $|x-y|^2 = (x-y)\cdot(x-y) =$

$$|x|^2 + |y|^2 + 2x\cdot y$$

$$\leq |x|^2 + |y|^2 + 2|x||y|$$

$$= \left(|x| + |y|\right)^2$$ & e) is proved.

f) follows from e)

Sets and functions:

$f : A \longrightarrow B$

A — set, B — set

f — rule that associates exactly one element of B to every element of A.

$\{f(x) : x \in A\} = \text{Range}(f)$

$A = \text{Domain}(f)$

If $E \subset A$, $\{f(x) : x \in E\} = f(E)$

is called the image of E under f.

If $f(A) = B$, f is called onto.

Inverse image: If $E \subseteq B$,

$$f^{-1}(E) = \{x \in A : f(x) \in E\}$$

i.e inverse image of E under f.